

# Minimal Skew energy of oriented bicyclic graphs with a given diameter \*

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## Abstract

Let  $S(G^\sigma)$  be the skew-adjacency matrix of the oriented graph  $G^\sigma$ , which is obtained from a simple undirected graph  $G$  by assigning an orientation  $\sigma$  to each of its edges. The skew energy of an oriented graph  $G^\sigma$  is defined as the sum of absolute values of all eigenvalues of  $S(G^\sigma)$ . For any positive integer  $d$  with  $3 \leq d \leq n-3$ , we determine the graph with minimal skew energy among all oriented bicyclic graphs that contain no vertex disjoint odd cycle of lengths  $s$  and  $l$  with  $s+l \equiv 2(mod 4)$  on  $n$  vertices with a given diameter  $d$ .

**Key Words:** Oriented graph, Bicyclic graph, Skew energy, Diameter.

**AMS Subject Classification (1991):** 05C50, 15A18.

## 1 Introduction

Let  $G$  be a simple undirected graph with an orientation  $\sigma$ , which assigns to each edge a direction so that  $G^\sigma$  becomes an oriented graph. Then  $G$  is usually called the underlying graph of  $G^\sigma$ . The skew-adjacency matrix of  $G^\sigma$  with vertex set  $V(G) = \{1, 2, \dots, n\}$  is the  $n \times n$  matrix  $S(G^\sigma) = [s_{ij}]$ ,  $s_{ij} = 1$  and  $s_{ji} = -1$  if  $(i, j)$  is an arc of  $G^\sigma$ , and  $s_{ij} = s_{ji} = 0$  otherwise. Since  $S(G^\sigma)$  is a real skew symmetric matrix, all eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$  of  $S(G^\sigma)$  are pure imaginary numbers or 0.

The skew energy of an oriented graph  $G^\sigma$ , denoted by  $E_s(G^\sigma)$ , is defined as the sum of absolute values of all eigenvalues of  $S(G^\sigma)$  (see [1]), that is

$$E_s(G^\sigma) = \sum_{i=1}^n |\lambda_i|,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of the skew-adjacency matrix  $S(G^\sigma)$ , namely the  $n$  roots of  $\phi(G^\sigma; x) = 0$ . Here  $\phi(G^\sigma; x) = \det(xI_n - S(G^\sigma)) = \sum_{i=0}^n a_i(G^\sigma)x^{n-i}$  is the skew

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characteristic polynomial of  $G^\sigma$ , where  $I_n$  is the unit matrix of order  $n$ . Since  $S(G^\sigma)$  is a real skew symmetric matrix,  $a_{2i}(G^\sigma) \geq 0$  and  $a_{2i+1}(G^\sigma) = 0$  for all  $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$  (see [1]). So we have

$$\phi(G^\sigma; x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}(G^\sigma) x^{n-2i}. \quad (1)$$

By using the coefficients of  $\phi(G^\sigma; x)$ , the skew energy  $E_s(G^\sigma)$  can be expressed by the following integral formula [5]

$$E_s(G^\sigma) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[ \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}(G^\sigma) x^{2i} \right] dx. \quad (2)$$

It follows that  $E_s(G^\sigma)$  is a strictly monotonously increasing function of  $a_{2i}(G^\sigma)$  for  $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$  for any oriented graph. Note that  $a_0(G^\sigma) = 1$  and  $a_2(G^\sigma)$  equals to the number of the edges in  $G$ . This provides a useful way for comparing the skew energies of a pair of oriented graphs.

Let  $G_1^{\sigma_1}$  and  $G_2^{\sigma_2}$  be two oriented graphs of order  $n$ . If  $a_{2i}(G_1^{\sigma_1}) \leq a_{2i}(G_2^{\sigma_2})$  for all  $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ , we write  $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$ . Furthermore, if  $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$  and there exists at least one index  $j$  such that  $a_{2j}(G_1^{\sigma_1}) < a_{2j}(G_2^{\sigma_2})$ , we write  $G_1^{\sigma_1} \prec G_2^{\sigma_2}$ . If  $a_{2i}(G_1^{\sigma_1}) = a_{2i}(G_2^{\sigma_2})$  for all  $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ , we write  $G_1^{\sigma_1} \sim G_2^{\sigma_2}$ . According to the integral formula (2), we have

$$\begin{aligned} G_1^{\sigma_1} \preceq G_2^{\sigma_2} &\Rightarrow E_s(G_1^{\sigma_1}) \leq E_s(G_2^{\sigma_2}); \\ G_1^{\sigma_1} \prec G_2^{\sigma_2} &\Rightarrow E_s(G_1^{\sigma_1}) < E_s(G_2^{\sigma_2}). \end{aligned}$$

The study on the extremal values of energy for oriented graphs is of importance for the chemical graph theory, and a lot of interesting results have been reported. For the oriented unicyclic graphs of order  $n$ , Hou et al. [5] obtained the oriented graphs with the 1st-minimal, the 2nd-minimal and the maximal skew energies, and Zhu [18] determined the oriented graphs with the first  $\lfloor \frac{n-9}{2} \rfloor$  largest skew energies. For the oriented bicyclic graphs, Shen et al. [11] deduced the oriented graphs with the minimal and maximal skew energies, and Wang et al. [13] characterized the oriented graph with the second largest skew energy. Zhu and Yang [19] obtained the oriented unicyclic graphs that have perfect matchings with the minimal skew energy. Yang et al. [15] determined the oriented unicyclic graphs of a fixed diameter with the minimal skew energy. Some other results about the extremal skew energies can be found in Refs. [12, 2, 8]. For a survey on skew energy of oriented graphs, one can refer to [9].

This paper is organized as follows: In Section 2, we give some notations and preliminary results, which will be used in the following discussion. The graph with minimal skew energy among all oriented bicyclic graphs that contain no vertex disjoint odd cycle of lengths  $s$  and  $l$  with  $s+l \equiv 2(mod 4)$  on  $n$  vertices with a given diameter  $d$  will be determined in Section 3, where  $3 \leq d \leq n-3$ . For  $d=2$ , we can refer to [11].

## 2 Preliminary Results

Let  $G = (V(G), E(G))$  be a simple graph. Denote by  $G - e$  the graph obtained from  $G$  by deleting the edge  $e$  and by  $G - v$  the graph obtained from  $G$  by deleting the vertex  $v$  together with all edges incident to it. Let  $d(G)$  be the diameter of  $G$ , which is defined as the greatest distance between any two vertices in  $G$ . The union of the graphs  $G_1 = (V(G_1), E(G_1))$  and  $G_2 = (V(G_2), E(G_2))$ , denoted by  $G_1 \cup G_2$ , is the graph with vertex set  $V(G_1) \cup V(G_2)$  and

edge set  $E(G_1) \cup E(G_2)$ .  $N(u)$  denotes the neighborhood of  $u$ . We refer to Cvetković et al. [3] for undefined terminology and notation.

For convenience, in terms of defining subgraph, matching, degree, diameter, etc., of an oriented graph, we focus only on its underlying graph. Moreover, we will briefly use the notations  $S_n$ ,  $P_n$  and  $C_n$  to denote the oriented star, the oriented path and the oriented cycle on  $n$  vertices, respectively, if no conflict exists there.

Let  $C$  be an even cycle of  $G$ . Then we say  $C$  is evenly oriented relative to  $G^\sigma$  if it has even number of edges oriented in the direction of the routing, otherwise  $C$  is oddly oriented. A linear subgraph  $L$  of  $G$  is a disjoint union of some edges and some cycles in  $G$ . A linear subgraph  $L$  is called evenly linear subgraph if the number of vertices of  $L$  is even.  $\varepsilon\mathcal{L}_i$  denotes the set of all evenly linear subgraph of  $G$  with  $i$  vertices.

Let  $G^\sigma$  be an oriented graph of  $G$ . Let  $W$  be a subset of  $V(G)$  and  $\overline{W} = V(G) \setminus W$ . The orientation  $G^\tau$  of  $G$  obtained from  $G^\sigma$  by reversing the orientations of all arcs between  $W$  and  $\overline{W}$ . Then  $G^\tau$  is said to be obtained from  $G^\sigma$  by a switching with respect to  $W$ . Moreover, two orientations  $G^\sigma$  and  $G^\tau$  of a graph  $G$  are said to be switching equivalent if  $G^\tau$  can be obtained from  $G^\sigma$  by a sequence of switchings. As noted in [1], since the skew-adjacency matrices obtained by a switching are similar, their skew energies are equal.

It is easy to verify that up to switching equivalence there are just two orientations of a cycle  $C$ : (1) Just one edge on the cycle has the opposite orientation to that of others, we denote this orientation by  $+$ . (2) All edges on the cycle  $C$  have the same orientation, we denote this orientation by  $-$ . So if a cycle is of even length and oddly oriented, then it is equivalent to the orientation  $+$ . If a cycle is of even length and evenly oriented, then it is equivalent to the orientation  $-$ .

Adiga et al. [1] showed that the skew energy of a directed tree is independent of its orientation, which is equal to the energy of its underlying tree. So by switching equivalence, for a unicyclic oriented graph or bicyclic oriented graph, we only need to consider the orientations of cycles.

Let  $C_a, C_b$  be two cycles in bicyclic graph  $G$  with  $t(t \geq 0)$  common vertices. If  $t \leq 1$ , then  $G$  contains exactly two cycles. If  $t \geq 2$ , then  $G$  contains exactly three cycles. The third cycle is denoted by  $C_c$ , where  $c = a + b - 2t + 2$ . Let  $C_a = v_0v_1 \cdots v_{a-1}v_0$  and  $C_b = u_0u_1 \cdots u_{b-1}u_0$ . If  $C_a$  and  $C_b$  have no common vertices, then  $C_a$  and  $C_b$  are connected by a path  $P$ , say from  $v_0$  to  $u_0$ . Let  $l(G)$  be the length of  $P$ . If  $t \geq 1$ ,  $C_c = u_0u_{b-1} \cdots u_tu_{t-1}v_tv_{t+1} \cdots v_{a-1}v_0$  is the third cycle, where  $v_0 = u_0$ ,  $v_1 = u_1, \dots, v_{t-1} = u_{t-1}$ . If we write  $w_0 = u_0, w_1 = u_{b-1}, \dots, w_{c-1} = v_{a-1}$ , then  $C_c = w_0w_1 \cdots w_{c-1}w_0$ .

For convenience, we denote by  $G^+$  (resp.  $G^-$ ) the unicyclic graph on which the orientation of a cycle is of orientation  $+$  (resp.  $-$ ), and denote by  $G^*$  the unicyclic graph on which the orientation of a cycle is of arbitrary orientation  $*$ . If  $t \leq 1$ , we denote by  $G^{\alpha,\beta}$  the bicyclic graph on which cycle  $C_a$  is of orientation  $\alpha$  and cycle  $C_b$  is of orientation  $\beta$ , where  $\alpha, \beta \in \{+, -, *\}$ . If  $t \geq 2$ , we denote by  $G^{\alpha,\beta,\gamma}$  the bicyclic graph on which  $C_a$  is of orientation  $\alpha$ ,  $C_b$  is of orientation  $\beta$  and  $C_c$  is of orientation  $\gamma$ , where  $\alpha, \beta, \gamma \in \{+, -, *\}$ .

The following results are the cornerstone of our discussion below, which gives an interpretation of all coefficients of the skew characteristic polynomial of an oriented graph.

**Lemma 2.1.** ([4]) *Let  $G^\sigma$  be an oriented graph of a graph  $G$  with the skew characteristic polynomial  $\phi(G^\sigma; x) = \sum_{i=0}^n a_i(G^\sigma)x^{n-i}$ . Then*

$$a_i(G^\sigma) = \sum_{L \in \mathcal{L}_i} (-2)^{p_e(L)} 2^{p_o(L)},$$

where  $p_e(L)$  (resp.  $p_o(L)$ ) is the number of all evenly (resp. oddly) oriented cycles of a linear subgraph  $L$  relative to  $G^\sigma$ .

**Lemma 2.2.** ([4]) Let  $e = (u, v)$  be an arc of an oriented graph  $G^\sigma$ . Then

$$\begin{aligned} a_i(G^\sigma) = & a_i(G^\sigma - e) + a_{i-2}(G^\sigma - u - v) + 2 \sum_{e \in C \in \text{Od}(G^\sigma)} a_{i-|V(C)|}(G^\sigma - V(C)) \\ & - 2 \sum_{e \in C \in \text{Ev}(G^\sigma)} a_{i-|V(C)|}(G^\sigma - V(C)), \end{aligned}$$

where  $\text{Od}(G^\sigma)$  (resp.  $\text{Ev}(G^\sigma)$ ) denotes the set of all oddly (resp. evenly) cycles of  $G^\sigma$ .

**Lemma 2.3.** ([14]) Let  $v$  be a vertex of an oriented graph  $G^\sigma$ . Then

$$\begin{aligned} a_i(G^\sigma) = & a_i(G^\sigma - v) + \sum_{u \in N(v)} a_{i-2}(G^\sigma - u - v) + 2 \sum_{v \in C \in \text{Od}(G^\sigma)} a_{i-|V(C)|}(G^\sigma - V(C)) \\ & - 2 \sum_{v \in C \in \text{Ev}(G^\sigma)} a_{i-|V(C)|}(G^\sigma - V(C)), \end{aligned}$$

where  $\text{Od}(G^\sigma)$  (resp.  $\text{Ev}(G^\sigma)$ ) denotes the set of all oddly (resp. evenly) cycles of  $G^\sigma$ .

From the Lemma 2.2, we can obtain easily Lemmas 2.4 and 2.5.

**Lemma 2.4.** Let  $e$  be a cut edge of  $G$ . Then  $G^\sigma \succeq G^\sigma - e$

**Lemma 2.5.** Let  $G$  be a unicyclic graph with  $n$  vertices or a bicyclic graph with  $n$  vertices. Then  $G^\sigma \succeq S_n$ .

Let  $\mathcal{T}(n, d)$  be the class of trees with  $n$  vertices and diameter  $d$ . Denote by  $T_{n,d}$  the tree obtained from the path  $P_{d-1}$  and the star  $S_{n-d+2}$  by identifying one pendent vertex of them.  $\mathcal{U}(n, d)$  denotes the class of unicyclic graphs with  $n$  vertices and diameter  $d$  and  $U_{n,d}$  denotes the unicyclic graph obtained from the cycle  $C_4$  by attaching a pendent vertex of the path  $P_{d-2}$  and  $n - d - 1$  pendent edges to its two non-adjacent vertices respectively; see Figure 1.

Let  $\mathcal{B}(n)$  be the class of bicyclic graphs with  $n$  vertices and contains no vertex disjoint odd cycles of lengths  $s$  and  $l$  with  $s + l \equiv 2(\text{mod} 4)$ . Let  $\mathcal{B}(n, d)$  be the class of bicyclic graphs  $\mathcal{B}(n)$  with diameter  $d$  where  $2 \leq d \leq n - 3$ . Denote by  $B_{n,d}$  the bicyclic graph obtained from the  $K_{2,3}$  by attaching a pendent vertex of the path  $P_{d-2}$  and  $n - d - 2$  pendent edges to its two vertices of degree three respectively; see Figure 1.

We have known that the skew energy of a directed tree is independent of its orientation. Hence, the following results for undirected trees apply equally well to oriented trees, which will be cited in the following discussion directly.

**Lemma 2.6.** ([7]) For  $n \geq 2$ ,  $P_n \succeq P_i \cup P_{n-i} \succeq P_1 \cup P_{n-1}$ .

**Lemma 2.7.** ([6]) Let  $n \geq 5$ ,  $T_n$  denote any tree with order  $n$  and  $T_n \neq P_n, S_n$ . Then  $P_n \succeq T_n \succeq S_n$ .

**Lemma 2.8.** ([16]) Let  $T \in \mathcal{T}(n, d)$ . Then  $T \succeq T_{n,d}$ .

**Lemma 2.9.** ([10]) If  $d > d_0 \geq 3$ , then  $T_{n,d} \succeq T_{n,d_0}$ .

**Lemma 2.10.** ([17]) If  $2 \leq d_1 < n_1 - 2$ , then  $T_{n_1,d_1} \cup T \succeq T_{n_1+n_2-1,d_1+d_2}$ , where  $T = T_{n_2,d_2}$  if  $2 \leq d_2 < n_2 - 1$  or  $P_2$  if  $n_2 = 2$  and  $d_2 = 1$ .

**Lemma 2.11.** ([15]) Let  $U \in \mathcal{U}(n, d)$  with  $n \geq 6$  and  $3 \leq d \leq n - 2$ . Then  $U^\sigma \succeq U_{n,d}^-$ .

**Lemma 2.12.** For  $3 \leq d \leq n - 2$ ,  $U_{n,d}^- \succeq T_{n,d}$ .

**Proof.** By Lemma 2.2,

$$\begin{aligned} a_{2i}(U_{n,d}^-) &= a_{2i}(T_{n,d}) + a_{2i-2}(P_{d-3} \cup S_{n-d+1}) - 2a_{2i-4}(P_{d-3}) \\ &= a_{2i}(T_{n,d}) + a_{2i-2}(P_{d-3} \cup S_{n-d-1}) \geq a_{2i}(T_{n,d}). \end{aligned}$$

■

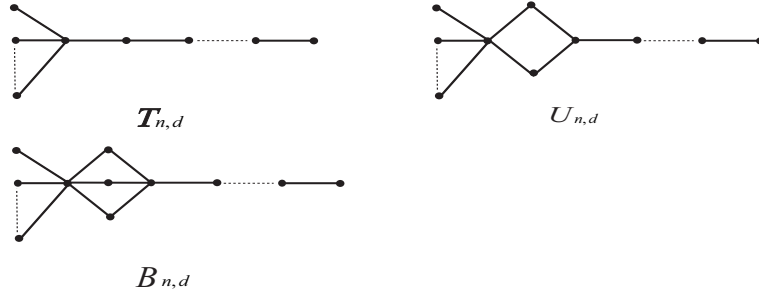


Figure 1: The tree  $T_{n,d}$ , the unicyclic graph  $U_{n,d}$  and the bicyclic graph  $B_{n,d}$ .

**Lemma 2.13.** If  $3 \leq d_0 < d \leq n - 2$ , then  $U_{n,d}^- \succeq U_{n,d_0}^-$ .

**Proof.** If  $d = 4$ , we have  $U_{n,4}^- \succeq U_{n,3}^-$  by Lemma 2.1. If  $d \geq 5$ , by Lemmas 2.2, 2.4 and 2.12,

$$\begin{aligned} a_{2i}(U_{n,d}^-) &= a_{2i}(U_{n-1,d-1}^-) + a_{2i-2}(U_{n-2,d-2}^-) \\ &\geq a_{2i}(U_{n-1,d-1}^-) + a_{2i-2}(T_{n-2,d-2}) \\ &\geq a_{2i}(U_{n-1,d-1}^-) + a_{2i-2}(T_{d-1,d-3}) \\ &= a_{2i}(U_{n,d-1}^-), \end{aligned}$$

so  $U_{n,d}^- \succeq U_{n,d-1}^- \succeq \dots \succeq U_{n,d_0}^-$ .

Similarly, we have the following result.

■

**Lemma 2.14.** If  $3 \leq d_0 < d \leq n - 2$ , then  $B_{n,d}^{-,-,-} \succeq B_{n,d_0}^{-,-,-}$ .

### 3 Main Results

**Lemma 3.1.** *Let  $G \in \mathcal{B}(n, n-3)$  with  $n \geq 6$ , and  $G \neq B_{n,n-3}$ . Then  $G^\sigma \succ B_{n,n-3}^{\bar{-}, \bar{-}, \bar{-}}$ .*

**Proof.** We prove this lemma by induction on  $n$ .

When  $n = 6$ , then  $G \in \mathcal{B}(6, 3)$  and  $G \neq B_{6,3}$ . Then  $G$  is isomorphic to one of the graphs in Figure 2. By Lemma 2.1, we have  $G^\sigma \succ B_{6,3}^{\bar{-}, \bar{-}, \bar{-}}$ .

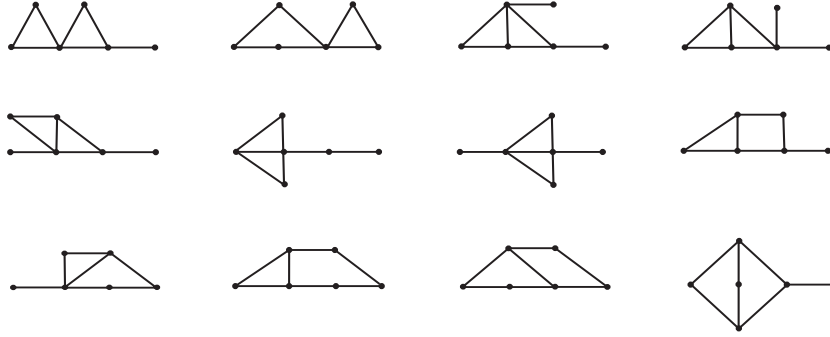


Figure 2: Graphs in  $\mathcal{B}(6, 3)$  except  $B_{6,3}$ .

When  $n = 7$ , then  $G \in \mathcal{B}(7, 4)$  and  $G \neq B_{7,4}$ . Then  $G$  is isomorphic to one of the graphs in Figure 3. By Lemma 2.1, we have  $\phi(B_{7,4}^{\bar{-}, \bar{-}, \bar{-}}; x) = x^7 + 8x^5 + 7x^3$ . By a directly calculation, we have  $a_4(G^\sigma) > a_4(B_{7,4}^{\bar{-}, \bar{-}, \bar{-}}) = 7$ . So  $G^\sigma \succ B_{7,4}^{\bar{-}, \bar{-}, \bar{-}}$ .

Suppose that the result holds for graphs of  $\mathcal{B}(n-1, n-4)$  and  $\mathcal{B}(n-2, n-5)$  with  $n \geq 8$ . Now suppose that  $G \in \mathcal{B}(n, n-3)$  and  $G \neq B_{n,n-3}$ .

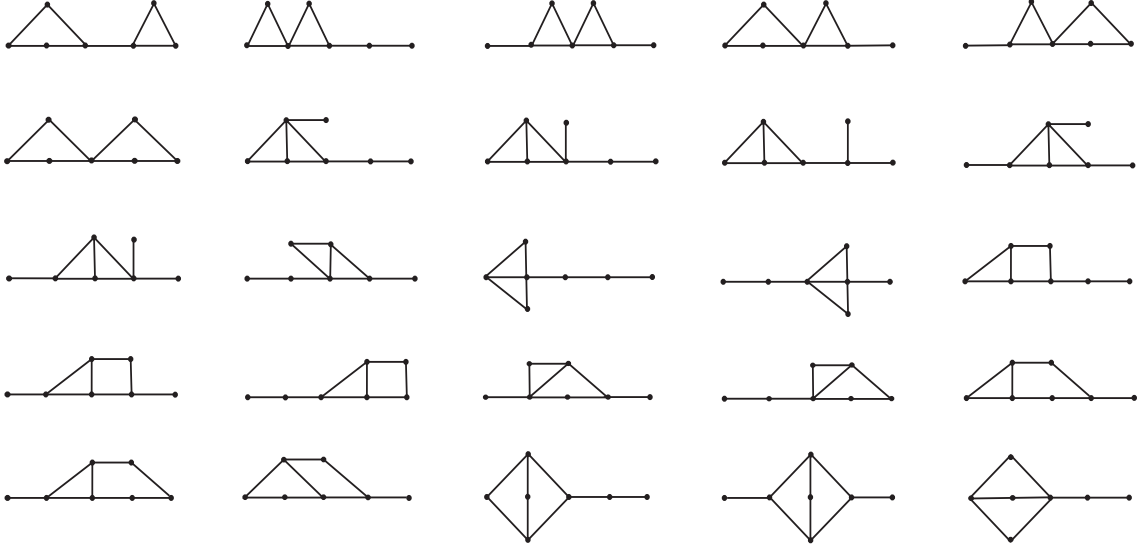


Figure 3: Graphs in  $\mathcal{B}(7, 4)$  except  $B_{7,4}$ .

**Case 1.** There exists a pendent vertex  $u$  in  $G$  such that the degree of its neighbor  $v$  is two. Then  $G - u \in \mathcal{B}(n-1, n-4)$  and  $G - u - v \in \mathcal{B}(n-2, n-5)$ . By Lemma 2.2, we have

$$a_{2i}(G^\sigma) = a_{2i}(G^\sigma - u) + a_{2i-2}(G^\sigma - u - v),$$

$$a_{2i}(B_{n,n-3}^{-,-,-}) = a_{2i}(B_{n-1,n-4}^{-,-,-}) + a_{2i-2}(B_{n-2,n-5}^{-,-,-}).$$

Note that  $G \neq B_{n,n-3}$ , thus  $G - u \neq B_{n-1,n-4}$  and  $G - u - v \neq B_{n-2,n-5}$ . Combining with the induction hypothesis, then  $G^\sigma \succ B_{n,n-3}^{-,-,-}$ .

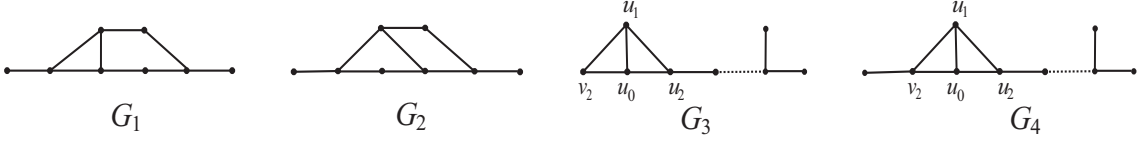


Figure 4: Graphs  $G_j$   $j=1,2,3,4$ .

**Case 2.** The neighbor of any pendent vertex has degree at least three or there is no pendent vertex. Then  $G$  is isomorphic to some  $G_j$  in Figure 4,  $j = 1, 2, 3, 4$ , or  $G$  contains one quadrangle which has at most one common vertex with another cycle that is a triangle or a quadrangle. For  $n = 8$ , if  $G$  is isomorphic to  $G_1$  or  $G_2$ , then  $G^\sigma \succ B_{8,5}^{-,-,-}$  by a directly calculation.

**Subcase 2.1.**  $G$  is isomorphic to  $G_3$  or  $G_4$ , then by Lemmas 2.2 and 2.8, we have

$$\begin{aligned} a_{2i}(G^\sigma) &= a_{2i}(G^\sigma - u_0 u_2) + a_{2i-2}(G^\sigma - u_0 - u_2) - 2a_{2i-4}(G^\sigma - u_0 - u_1 - u_2 - v_2) \\ &= a_{2i}(G^\sigma - u_0 u_2 - u_0 v_2) + a_{2i-2}(G^\sigma - u_0 - v_2) + a_{2i-2}(G^\sigma - u_0 - u_2) \\ &\quad - 2a_{2i-4}(G^\sigma - u_0 - u_1 - u_2 - v_2) \\ &= a_{2i}(G^\sigma - u_0 u_2 - u_0 v_2) + a_{2i-2}(G^\sigma - u_0 - v_2 - u_1 u_2) \\ &\quad + a_{2i-2}(G^\sigma - u_0 - u_2 - u_1 v_2) \\ &\geq a_{2i}(T_{n,n-3}) + 2a_{2i-2}(P_{n-6}) = a_{2i}(B_{n,n-3}^{-,-,-}), \end{aligned}$$

thus  $G^\sigma \succ B_{n,n-3}^{-,-,-}$ .

**Subcase 2.2.**  $G$  contains one quadrangle which has at most one common vertex with another cycle that is a triangle or a quadrangle. If  $n = 8, 9$ , it can be checked by Lemma 2.1 that  $G^\sigma \succ B_{n,n-3}^{-,-,-}$ . If  $n \geq 10$ , suppose that  $C_b = u_0 u_1 u_2 u_3 u_0$  (see Section 2) is a quadrangle. By Lemmas 2.2, 2.11 and 2.12, we have

$$\begin{aligned} a_{2i}(G^\sigma) &= a_{2i}(G^\sigma - u_0 u_1) + a_{2i-2}(G^\sigma - u_0 - u_1) - 2a_{2i-4}(G^\sigma - u_0 - u_1 - u_2 - u_3) \\ &= a_{2i}(G^\sigma - u_0 u_1 - u_1 u_2) + a_{2i-2}(G^\sigma - u_1 - u_2) + a_{2i-2}(G^\sigma - u_0 - u_1) \\ &\quad - 2a_{2i-4}(G^\sigma - u_0 - u_1 - u_2 - u_3) \\ &= a_{2i}(G^\sigma - u_0 u_1 - u_1 u_2) + a_{2i-2}(G^\sigma - u_1 - u_2 - u_0 u_3) \\ &\quad + a_{2i-2}(G^\sigma - u_0 - u_1 - u_2 u_3) \\ &\geq a_{2i}(U_{n-1,n-3}^-) + a_{2i-2}(U_{n-3,n-5}^-) + a_{2i-2}(U_{n-4,n-6}^-) \\ &\geq a_{2i}(U_{n-1,n-3}^-) + a_{2i-2}(U_{n-5,n-7}^- \cup P_2) + a_{2i-2}(U_{n-4,n-6}^-). \end{aligned}$$

By Lemma 2.2, we have  $a_{2i}(B_{n,n-3}^{-,-,-}) = a_{2i}(U_{n-1,n-3}^-) + a_{2i-2}(P_{n-6} \cup P_2) + a_{2i-2}(P_{n-5})$ , so  $G^\sigma \succ B_{n,n-3}^{-,-,-}$ . ■

**Lemma 3.2.** Let  $G \in \mathcal{B}(n, d)$  with  $3 \leq d \leq n - 4$ . If  $G$  contains no pendent vertices, then  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Proof.** Let  $b \geq a$ . Since  $d \leq n - 4$ , we have  $b \geq 5$ . By Lemma 2.2, we have

$$a_{2i}(B_{n,d+1}^{-,-,-}) = a_{2i}(U_{n,d+1}^{-}) + a_{2i-2}(P_{d-2} \cup S_{n-d}) - 4a_{2i-4}(P_{d-2}). \quad (3)$$

$$a_{2i}(B_{n,d+1}^{-,-,-}) = a_{2i}(U_{n-1,d+1}^{-}) + a_{2i-2}(P_{d-2} \cup S_{n-d-2}) + a_{2i-2}(P_{d-1}). \quad (\text{By (3)}) \quad (4)$$

$$a_{2i}(B_{n,d+1}^{-,-,-}) = a_{2i}(T_{n-1,d+1}) + a_{2i-2}(P_d) + 2a_{2i-2}(P_{d-2} \cup S_{n-d-3}). \quad (\text{By (4)}) \quad (5)$$

$$a_{2i}(B_{n,d+1}^{-,-,-}) = a_{2i}(T_{n,d+1}) + 2a_{2i-2}(P_{d-2} \cup S_{n-d-3}). \quad (\text{By (3)}) \quad (6)$$

**Case 1.** When  $t \leq 1$ , there are exactly two cycles  $C_a$  and  $C_b$  in  $G$ . Then  $d = \lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor + l(G)$  (see Section 2 for  $l(G)$ ).

**Subcase 1.1.** The length of  $C_b$  is odd. Then  $d(G - u_1u_2) = \lfloor \frac{a}{2} \rfloor + b + l(G) - 2 \geq d + 1$ ,  $d(G - u_1 - u_2) \geq d$  and  $d \geq 3$ . By Lemmas 2.2, 2.11, 2.12 and 2.13, we have

$$\begin{aligned} a_{2i}(G^\sigma) &= a_{2i}(G^\sigma - u_1u_2) + a_{2i-2}(G^\sigma - u_1 - u_2) \\ &\geq a_{2i}(U_{n,d+1}^{-}) + a_{2i-2}(U_{n-2,d}^{-}) \\ &\geq a_{2i}(U_{n,d+1}^{-}) + a_{2i-2}(T_{n-2,d}^{-}). \end{aligned}$$

Combining with (3) and Lemma 2.14, then  $G^\sigma \succ B_{n,d+1}^{-,-,-} \succ B_{n,d}^{-,-,-}$ .

**Subcase 1.2.** The length of  $C_b$  is even. Then  $b \geq 6$ . Hence  $d(G - u_1u_2 - u_2u_3) = \lfloor \frac{a}{2} \rfloor + b + l(G) - 3 \geq d$ ,  $d(G - u_2 - u_3 - u_4u_5) \geq d - 2$ ,  $d(G - u_1 - u_2 - u_3u_4) \geq d - 1$ . and  $d \geq 4$ . If  $d = 4$ , it can be checked by Lemmas 2.1 that  $G^\sigma \succ B_{n,4}^{-,-,-}$ . If  $d \geq 5$ , by Lemmas 2.2, 2.11, 2.12 and 2.13, we have

$$\begin{aligned} a_{2i}(G^\sigma) &\geq a_{2i}(G^\sigma - u_1u_2 - u_2u_3) + a_{2i-2}(G^\sigma - u_1 - u_2) \\ &\quad + a_{2i-2}(G^\sigma - u_2 - u_3) - 2a_{2i-4}(G^\sigma - V(C_b)) \\ &\geq a_{2i}(G^\sigma - u_1u_2 - u_2u_3) + a_{2i-2}(G^\sigma - u_1 - u_2 - u_3u_4) \\ &\quad + a_{2i-2}(G^\sigma - u_2 - u_3 - u_4u_5) \\ &\geq a_{2i}(U_{n-1,d}^{-}) + a_{2i-2}(U_{n-3,d-1}^{-}) + a_{2i-2}(U_{n-3,d-2}^{-}) \\ &\geq a_{2i}(U_{n-1,d}^{-}) + a_{2i-2}(T_{n-3,d-1}) + a_{2i-2}(T_{n-3,d-2}). \end{aligned}$$

Combining with (4),  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Case 2.**  $t \geq 2$ . Note that  $a - t + 1 \geq t - 1$  and  $b - t + 1 \geq t - 1$ . Then  $c \geq b$  and  $d = \lfloor \frac{c}{2} \rfloor = \lfloor \frac{a+b}{2} \rfloor - t + 1$ .

**Subcase 2.1.**  $C_b$  and  $C_c$  are odd cycles. Then  $d(G - w_0w_1) = \lfloor \frac{a}{2} \rfloor + b - t \geq d + 1$ ,  $d(G - w_0 - w_1) \geq c - 3 \geq d$  and  $d \geq 3$ . By Lemmas 2.2, 2.8, 2.9, 2.11 and 2.13,

$$\begin{aligned} a_{2i}(G^\sigma) &= a_{2i}(G^\sigma - w_0w_1) + a_{2i-2}(G^\sigma - w_0 - w_1) \\ &\geq a_{2i}(U_{n,d+1}^{-}) + a_{2i-2}(T_{n-2,d}). \end{aligned}$$

Combining with (3) and Lemma 2.14, then  $G^\sigma \succ B_{n,d+1}^{-,-,-} \succ B_{n,d}^{-,-,-}$ .

**Subcase 2.2.**  $C_b$  is an odd cycle and  $C_c$  is an even cycle. If  $b = 5$ , we have  $G^\sigma \succ B_{n,d}^{-,-,-}$  by Lemma 2.1. Otherwise,  $d(G - w_0w_1 - w_1w_2) = \lfloor \frac{a}{2} \rfloor + b - t - 1 \geq d + 1$ ,  $d(G - w_1 - w_2 - w_3w_4) \geq$



$d-1$ ,  $d(G-w_0-w_1-w_2w_3) \geq c-4 \geq d$  and  $d \geq 4$ . By Lemmas 2.2, 2.8, 2.9, 2.11, 2.12 and 2.13, we have

$$\begin{aligned}
a_{2i}(G^\sigma) &\geq a_{2i}(G^\sigma - w_0w_1 - w_1w_2) + a_{2i-2}(G^\sigma - w_0 - w_1) \\
&\quad + a_{2i-2}(G^\sigma - w_1 - w_2) - 2a_{2i-c}(G^\sigma - V(C_c)) \\
&\geq a_{2i}(G^\sigma - w_0w_1 - w_1w_2) + a_{2i-2}(G^\sigma - w_0 - w_1 - w_2w_3) \\
&\quad + a_{2i-2}(G^\sigma - w_1 - w_2 - w_3w_4) \\
&\geq a_{2i}(U_{n-1,d+1}^-) + a_{2i-2}(U_{n-3,d-1}^-) + a_{2i-2}(T_{n-3,d}) \\
&\geq a_{2i}(U_{n-1,d+1}^-) + a_{2i-2}(T_{n-3,d-1}) + a_{2i-2}(T_{n-3,d}),
\end{aligned}$$

which, together with (4) and Lemma 2.14, implies  $G^\sigma \succ B_{n,d+1}^{-,-,-} \succ B_{n,d}^{-,-,-}$ .

For  $C_c$  is an odd cycle and  $C_b$  is an even cycle, we have  $G^\sigma \succ B_{n,d}^{-,-,-}$  by similar arguments as above.

**Subcase 2.3.**  $C_b$  and  $C_c$  are even cycles. Then  $c \geq b \geq 6$  and  $C_a$  is an even cycle.  $d(G-u_0) = c-2 \geq d+1$ ,  $d \geq 3$  and  $n = a+b-t$ . When  $a \neq 4$  and  $b \neq 6$ , we have  $n-d \geq 5$ . By Lemmas 2.3, 2.6, 2.7, 2.8 and 2.9, we have

$$\begin{aligned}
a_{2i}(G^\sigma) &\geq a_{2i}(G^\sigma - u_0) + a_{2i-2}(G^\sigma - u_0 - u_1) + a_{2i-2}(G^\sigma - u_0 - u_{b-1}) \\
&\quad + a_{2i-2}(G^\sigma - u_0 - v_{a-1}) - 2a_{2i-a}(G^\sigma - V(C_a)) \\
&\quad - 2a_{2i-b}(G^\sigma - V(C_b)) - 2a_{2i-c}(G^\sigma - V(C_c)) \\
&\geq a_{2i}(G^\sigma - u_0) + a_{2i-2}(G^\sigma - u_0 - u_1 - u_{t-1}) \\
&\quad + a_{2i-2}(G^\sigma - u_0 - u_{b-1} - u_{t-1}) + a_{2i-2}(G^\sigma - u_0 - v_{a-1} - u_{t-1}) \\
&\geq a_{2i}(T_{n-1,d+1}) + a_{2i-2}(P_{a-t} \cup P_{b-t-1} \cup P_{t-2}) + a_{2i-2}(P_{a-t} \cup P_{b-t} \cup P_{t-3}) \\
&\quad + a_{2i-2}(P_{a-t-1} \cup P_{b-t} \cup P_{t-2}) \quad (\text{By Lemma 2.6}) \\
&\geq a_{2i}(T_{n-1,d+1}) + 3a_{2i-2}(P_{a+b-t-5}) \quad (\text{By Lemma 2.7}) \\
&\geq a_{2i}(T_{n-1,d+1}) + a_{2i-2}(P_{n-5}) + 2a_{2i-2}(T_{n-5,d-1}).
\end{aligned}$$

Combining with (5) and Lemma 2.14,  $G^\sigma \succ B_{n,d+1}^{-,-,-} \succ B_{n,d}^{-,-,-}$ . When  $a = 4$  and  $b = 6$ , we have  $G^\sigma \succ B_{n,d}^{-,-,-}$  by Lemma 2.1. ■

**Lemma 3.3.** Let  $G \in \mathcal{B}(n, d)$  with  $3 \leq d \leq n-4$ . If  $G$  contains exactly one pendent vertex  $u$  on all diametrical paths of  $G$  such that  $G-u$  contains no pendent vertices, then  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Proof.** Let  $b \geq a$ . Since  $d \leq n-4$ , we have  $b \geq 5$ . Let  $v$  be the neighbor of  $u$ .

**Case 1.** When  $t \leq 1$ , there are exactly two cycles  $C_a$  and  $C_b$  in  $G$ . Then  $d = \lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor + l(G) + 1$ .

**Subcase 1.1.** The length of  $C_b$  is odd. If  $b \geq 7$ , then  $d(G-u_1u_2) \geq \lfloor \frac{a}{2} \rfloor + b + l(G) - 2 \geq d+1$ . If  $b = 5$  and  $v$  lies on  $C_a$ , then  $d(G-u_1u_2) \geq \lfloor \frac{a}{2} \rfloor + b + l(G) - 1 = d+1$ . In these cases, by similar arguments as those in Subcase 1.1 of Lemma 3.2,  $G^\sigma \succ B_{n,d}^{-,-,-}$ . Otherwise,  $a = 3, 4$ ,  $b = 5$ , and  $v$  lies on  $C_b$ . If  $t = 1$ , we have  $G^\sigma \succ B_{n,d}^{-,-,-}$  by Lemma 2.1. If  $t = 0$ , then  $G$  is isomorphic to  $G_5$  or  $G_6$  in Figure 5.

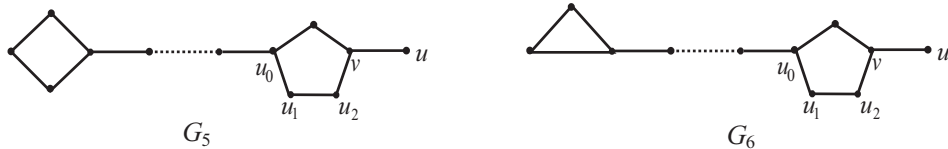


Figure 5: Graphs  $G_5$  and  $G_6$ .

By Lemmas 2.2, 2.11, 2.12 and 2.13, we have

$$\begin{aligned} a_{2i}(G^\sigma) &= a_{2i}(G^\sigma - u_1u_2) + a_{2i-2}(G^\sigma - u_1 - u_2) \\ &\geq a_{2i}(U_{n,d}^-) + a_{2i-2}(T_{n-2,d}), \end{aligned}$$

which, together with (3), implies  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Subcase 1.2.** The length of  $C_b$  is even. If  $b \geq 8$ , hence  $d(G - u_1u_2 - u_2u_3) = \lfloor \frac{a}{2} \rfloor + b + l(G) - 3 \geq d$ . If  $b = 6$  and  $v$  lies on  $C_a$ , then  $d(G - u_1u_2 - u_2u_3) = \lfloor \frac{a}{2} \rfloor + b + l(G) - 3 = d$ . In these cases, by similar arguments as those in Subcase 1.2 of Lemma 3.2,  $G^\sigma \succ B_{n,d}^{-,-,-}$ . If  $a = 5$ ,  $b = 6$ , and  $v$  lies on  $C_b$ , then  $G^\sigma \succ B_{n,d}^{-,-,-}$  by similar arguments as those in Subcase 1.1 of Lemma 3.2. Otherwise,  $a = 3, 4$ ,  $b = 6$ , and  $v$  lies on  $C_b$ . If  $l(G) \leq 2$ , by Lemma 2.1, then we have  $G^\sigma \succ B_{n,d}^{-,-,-}$ . If  $l(G) \geq 3$ , by Lemmas 2.2, 2.10, 2.11 and 2.12, we have

$$\begin{aligned} a_{2i}(G^\sigma) &\geq a_{2i}(G^\sigma - u_0u_1 - u_1u_2) + a_{2i-2}(G^\sigma - u_0 - u_1) \\ &\quad + a_{2i-2}(G^\sigma - u_1 - u_2) - 2a_{2i-b}(G^\sigma - V(C_b)) \\ &\geq a_{2i}(G^\sigma - u_0u_1 - u_1u_2) + a_{2i-2}(G^\sigma - u_0 - u_1 - u_2u_3) \\ &\quad + a_{2i-2}(G^\sigma - u_1 - u_2 - u_3u_4) \\ &\geq a_{2i}(U_{n-1,d}^-) + a_{2i-2}(U_{n-4,d-2}^- \cup P_2) + a_{2i-2}(U_{n-7,d-5}^- \cup P_4) \\ &\geq a_{2i}(U_{n-1,d}^-) + a_{2i-2}(T_{n-4,d-2} \cup P_2) + a_{2i-2}(T_{n-7,d-5} \cup P_4) \\ &\geq a_{2i}(U_{n-1,d}^-) + a_{2i-2}(T_{n-3,d-1}) + a_{2i-2}(T_{n-4,d-2}). \end{aligned}$$

Combining with (4),  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Case 2.**  $t \geq 2$ . Then  $d = \lfloor \frac{a}{2} \rfloor + 1 = \lfloor \frac{a+b}{2} \rfloor - t + 2$ . Since  $b \geq 5$ , assume that  $w_0, w_1 \neq v$ . Note that  $a - t + 1 \geq t - 1$  and  $b - t + 1 \geq t - 1$ . By similar arguments as those in Case 2 of Lemma 3.2, we have  $G^\sigma \succ B_{n,d}^{-,-,-}$ .  $\blacksquare$

**Theorem 3.4.** Let  $G \in \mathcal{B}(n, d)$  with  $3 \leq d \leq n-3$  and  $G \neq B_{n,d}$ . If  $t = 0$ , then  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Proof.** We prove this theorem by induction on  $n - d$ .

By Lemma 3.1, the result holds for  $n - d = 3$ . Let  $h \geq 4$  and suppose that the result holds for  $n - d < h$ . Now suppose that  $n - d = h$  and  $G \in \mathcal{B}(n, d)$ .

**Case 1.** There is no pendent vertex in  $G$ . By Lemma 3.2, we have  $G^\sigma \succ B_{n,d}^{-,-,-}$ .

**Case 2.** There is a pendent vertex  $u$  outside some diametrical path  $P(G) = x_0x_1 \cdots x_d$ . Let  $v$  be the neighbor of  $u$ . Then  $G - u \in \mathcal{B}(n - 1, d)$ . By the induction hypothesis,  $G^\sigma - u \succ B_{n-1,d}^{-,-,-}$ . By Lemma 2.2, we have

$$a_{2i}(B_{n,d}^{-,-,-}) = a_{2i}(B_{n-1,d}^{-,-,-}) + a_{2i-2}(T_{d+1,d-2}). \quad (7)$$

Let  $H = G - u - v$ , it suffices to prove that  $a_{2i}(H^\sigma) \geq a_{2i}(T_{d+1,d-2})$ .

**Subcase 2.1.**  $v$  lies on some cycle, say  $C_a$ .

**Subcase 2.1.1.** Suppose that  $P(G)$  and  $C_b$  have no common vertices. Then  $H \supseteq P_k \cup P_{d-k} \cup C_b$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$\begin{aligned} a_{2i}(H^\sigma) &\geq a_{2i}(P_k \cup P_{d-k} \cup C_b) \geq a_{2i}(P_{d-1} \cup S_b) \geq a_{2i}(P_{d-1} \cup P_3) \\ &\geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}). \end{aligned}$$

**Subcase 2.1.2.** Suppose that  $P(G)$  and  $C_b$  have common vertices  $x_l, \dots, x_{l+q}$ , where  $q \geq 0$ .

If  $v$  lies outside  $P(G)$ , then  $H \supseteq H_1$ , where  $H_1 \in U(s_1, d)$ ,  $s_1 \geq d + 2$ . By Lemmas 2.4, 2.7, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_1^\sigma) \geq a_{2i}(U_{s_1, d}^\sigma) \geq a_{2i}(T_{s_1, d}) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $v$  lies on  $P(G)$ . Then  $P(G)$  and  $C_a$  have common vertices  $x_k, \dots, x_{k+p}$ , where  $p \geq 0, k + p < l$ .

Suppose that  $p = 0$ , then  $k \geq 1$ ,  $H \supseteq P_2 \cup P_k \cup H_2$ , where  $H_2 \in U(s_2, d_2)$ ,  $s_2 \geq d_2 + 2$ ,  $d_2 \geq d - k - 1 \geq 1$ . If  $d_2 = 1$ , then  $k = d - 2$  and  $H_2 = C_3$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_2 \cup P_k \cup C_3) \geq a_{2i}(P_{d-1} \cup P_3) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $d_2 = 2$ , then  $k \geq d - 3$  and  $s_2 \geq 4$ . By Lemmas 2.4, 2.5, 2.6, 2.8 and 2.9, we have

$$\begin{aligned} a_{2i}(H^\sigma) &\geq a_{2i}(P_2 \cup P_k \cup H_2^\sigma) \geq a_{2i}(P_{k+1} \cup S_{s_2}) \geq a_{2i}(P_{k+1} \cup S_4) \\ &\geq a_{2i}(T_{k+4, k+2}) \geq a_{2i}(T_{d+1, d-1}) \geq a_{2i}(T_{d+1, d-2}). \end{aligned}$$

If  $d_2 \geq 3$ , then by Lemmas 2.4, 2.6, 2.8, 2.9, 2.11 and 2.12, we have

$$\begin{aligned} a_{2i}(H^\sigma) &\geq a_{2i}(P_2 \cup P_k \cup H_2^\sigma) \geq a_{2i}(P_{k+1} \cup U_{s_2, d_2}^-) \geq a_{2i}(P_{k+1} \cup T_{s_2, d_2}) \\ &\geq a_{2i}(T_{s_2+k, d_2+k}) \geq a_{2i}(T_{d_2+k+2, d_2+k}) \geq a_{2i}(T_{d+1, d-1}) \geq a_{2i}(T_{d+1, d-2}). \end{aligned}$$

Suppose that  $p \geq 1$ . If  $v = x_k$ , then  $H \supseteq P_k \cup H_3$ , where  $H_3 \in U(s_3, d_3)$ ,  $s_3 \geq d_3 + 2$ ,  $d_3 \geq d - k \geq 3$ . By Lemmas 2.4, 2.10, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup U_{s_3, d_3}^-) \geq a_{2i}(P_k \cup T_{s_3, d_3}) \geq a_{2i}(T_{s_3+k-1, d_3+k-1}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $v = x_{k+p}$ ,  $H \supseteq P_{k+p+1} \cup H_4$  or  $T_1 \cup H_4$  ( $p \geq 2$ ), where  $H_4 \in U(s_4, d_4)$ ,  $s_4 \geq d_4 + 2$ ,  $d_4 \geq d - k - p - 1 \geq 1$  and  $T_1 \in T(k + p + 1, k + p - 1)$ . If  $d_4 = 1$ , then  $k + p = d - 2$  and  $H_4 = C_3$ . By Lemmas 2.4, 2.5, 2.6, 2.7, 2.8, 2.9 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{d-1} \cup C_3) \geq a_{2i}(P_{d-1} \cup P_2) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1, d-2}).$$

or

$$a_{2i}(H^\sigma) \geq a_{2i}(T_{d-1, d-3} \cup C_3) \geq a_{2i}(T_{d-1, d-3} \cup P_3) \geq a_{2i}(T_{d+1, d-2}).$$

If  $d_4 = 2$ , then  $k + p \geq d - 3$  and  $s_4 \geq 4$ . By Lemmas 2.4, 2.5, 2.8, 2.9 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{k+p+1} \cup S_{s_4}) \geq a_{2i}(P_{k+p+1} \cup T_{4, 2}) \geq a_{2i}(T_{d+1, d-2}).$$

or

$$a_{2i}(H^\sigma) \geq a_{2i}(T_{k+p+1, k+p-1} \cup S_{s_4}) \geq a_{2i}(T_{k+p+1, k+p-1} \cup T_{4, 2}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $d_4 \geq 3$ , then by Lemmas 2.4, 2.8, 2.9, 2.10, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{k+p+1} \cup U_{s_4, d_4}^-) \geq a_{2i}(P_{k+p+1} \cup T_{s_4, d_4}) \geq a_{2i}(T_{d+1, d-2}).$$

or

$$a_{2i}(H^\sigma) \geq a_{2i}(T_{k+p+1, k+p-1} \cup U_{s_4, d_4}^-) \geq a_{2i}(T_{k+p+1, k+p-1} \cup T_{s_4, d_4}) \geq a_{2i}(T_{d+1, d-2}).$$

**Subcase 2.2.**  $v$  lies outside any cycle.

**Subcase 2.2.1.** Suppose that  $v$  lies on  $P(G)$  and  $v = x_k$ .

If  $P(G)$  and any cycle have no common vertices, then  $H \supseteq C_a \cup C_b \cup P_k \cup P_{d-k}$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(C_a \cup C_b \cup P_k \cup P_{d-k}) \geq a_{2i}(P_3 \cup P_{d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $P(G)$  and exactly one cycle have no common vertices, say  $C_a$ , then  $H \supseteq C_a \cup P_k \cup H_1$ , where  $H_1 \in U(s_1, d_1)$ ,  $s_1 \geq d_1 + 2$ ,  $d_1 \geq d - k - 1 \geq 1$ . If  $d_1 = 1$ , then  $k = d - 2$  and  $H_1 = C_3$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_2 \cup P_k \cup C_3) \geq a_{2i}(P_{d-1} \cup P_2) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_1 \geq 2$ , then  $k \geq d - 3$ . By Lemmas 2.4, 2.5, 2.6 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(C_a \cup P_k \cup H_1^\sigma) \geq a_{2i}(P_{k+1} \cup S_{s_1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $P(G)$  and two cycles have common vertices, then  $H \supseteq P_k \cup H_2$  or  $H_3 \cup H_4$ , where  $H_2 \in B(s_2, d_2)$ ,  $H_3 \in U(s_3, d_3)$ ,  $H_4 \in U(s_4, d_4)$ ,  $d_2 + 3 \leq s_2 \leq n - k - 2$ ,  $d_2 \geq d - k - 1 \geq 4$ ,  $s_3 \geq d_3 + 2$ ,  $d_3 \geq k - 1 \geq 2$ ,  $s_4 \geq d_4 + 2$ ,  $d_4 \geq d - k - 1 \geq 1$ .

Suppose that  $H \supseteq P_k \cup H_2$ ,  $s_2 - d_2 < h$  and  $d_2 \geq 4$ , by the induction hypothesis,  $H_2^\sigma \succ B_{s_2,d_2}$ . By Lemmas 2.4 and 2.10, and (6), we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup H_2^\sigma) \geq a_{2i}(P_k \cup B_{s_2,d_2}^{\sigma,-,-}) \geq a_{2i}(P_k \cup T_{s_2,d_2}) \geq a_{2i}(T_{d+1,d-2}).$$

Suppose that  $H \supseteq H_3 \cup H_4$ . If  $d_3 = 2$  and  $d_4 = 1$ , then  $d = 5$  and  $s_3 \geq 4$ . By Lemmas 2.4, 2.5, 2.9 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_3^\sigma \cup H_4^\sigma) \geq a_{2i}(S_{s_3} \cup C_3^-) \geq a_{2i}(P_3 \cup T_{4,2}) \geq a_{2i}(T_{6,4}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_3 = 2$  and  $d_4 = 2$ , then  $d = 6$  and  $s_3 \geq 4$ ,  $s_4 \geq 4$ . By Lemmas 2.4, 2.5 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_3^\sigma \cup H_4^\sigma) \geq a_{2i}(S_{s_3} \cup S_{s_4}) \geq a_{2i}(T_{4,2} \cup T_{4,2}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_3 \geq 3$  and  $d_4 = 1$ , then  $d_3 \geq d - 3$ . By Lemmas 2.4, 2.5, 2.9, 2.10, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_3^\sigma \cup H_4^\sigma) \geq a_{2i}(U_{s_3,d_3}^- \cup C_3^-) \geq a_{2i}(T_{s_3,d_3} \cup P_3) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_3 \geq 3$  and  $d_4 = 2$ , then  $d_3 \geq d - 4$ ,  $s_4 \geq 4$ . By Lemmas 2.4, 2.5, 2.10, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_3^\sigma \cup H_4^\sigma) \geq a_{2i}(U_{s_3,d_3}^- \cup S_{s_4}) \geq a_{2i}(T_{s_3,d_3} \cup T_{4,2}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_3 \geq 3$  and  $d_4 \geq 3$ , then  $d_3 + d_4 \geq d - 2$ . By Lemmas 2.4, 2.10, 2.11 and 2.12, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_3^\sigma \cup H_4^\sigma) \geq a_{2i}(U_{s_3,d_3}^- \cup U_{s_4,d_4}^-) \geq a_{2i}(T_{s_3,d_3} \cup T_{s_4,d_4}) \geq a_{2i}(T_{d+1,d-2}).$$

**Subcase 2.2.2.** Suppose that  $v$  lies outside  $P(G)$ . Then  $G \supseteq C_a \cup C_b \cup P(G)$ ,  $C_a \cup H_1$  or  $H_2$ , where  $H_1 \in U(s_1, d)$  with  $s_1 \geq d + 2$  and  $H_2 \in B(s_2, d)$  with  $d + 3 \leq s_2 \leq n - 2$ . We can prove  $a_{2i}(H^\sigma) \geq a_{2i}(T_{d+1,d-2})$  by similar arguments as above.

**Case 3.** All pendent vertices are contained in the  $P(G)$ , where  $P(G) = x_0 x_1 \cdots x_d$  is a diametrical path of  $G$ . Suppose that  $y_0 y_1 \cdots y_p$  is a path whose internal vertices  $y_1, y_2, \dots, y_{p-1}$

all have degree two and  $y_p$  is a pendent vertex. Then we say that it is a pendent path, denoted by  $(y_0, y_p)$  (see [17]).

**Subcase 3.1.** There are exactly two pendent vertices,  $x_0$  and  $x_d$ . Suppose that  $\deg(x_k)$ ,  $\deg(x_l) \geq 3$  and that  $(x_k, x_0)$  and  $(x_l, x_d)$  are distinct pendent paths. Let  $s = k - l$ .

If  $s = 0$  ( $x_k = x_l$ ), then  $k \geq 3$  and  $l < d - 3$ . So it suffices to prove that  $H_1^\sigma, H_2^\sigma \succ B_{n-d+3,3}^{\bar{-},\bar{-},\bar{-}}, H_3^\sigma, H_4^\sigma \succ S_{n-d+2}$ , where  $H_1 = G - \{x_{k-3}, \dots, x_0\} - \{x_{l+2}, \dots, x_d\}$ ,  $H_3 = G - \{x_{k-3}, \dots, x_0\} - \{x_{l+1}, \dots, x_d\}$ ,  $H_2 = G - \{x_{k-2}, \dots, x_0\} - \{x_{l+3}, \dots, x_d\}$ , and  $H_4 = G - \{x_{k-2}, \dots, x_0\} - \{x_{l+2}, \dots, x_d\}$ . By Lemma 2.5,  $H_3^\sigma, H_4^\sigma \succ S_{n-d+2}$ . Let  $d_1 = d(H_1)$ . Since  $d_1 \geq 4, n - d + 3 - d_1 < h$ .  $H_1^\sigma \succ B_{n-d+3,d_1}^{\bar{-},\bar{-},\bar{-}} \succ B_{n-d+3,3}^{\bar{-},\bar{-},\bar{-}}$  by the induction hypothesis and Lemma 2.14. Similarly,  $H_2^\sigma \succ B_{n-d+3,3}^{\bar{-},\bar{-},\bar{-}}$ .

If  $s = 1$  or  $2$ , then by similar arguments as above, we have  $G^\sigma \succ B_{n,d}^{\bar{-},\bar{-},\bar{-}}$ .

If  $s \geq 3$ , then we only need consider the case  $k \geq 2$  and  $l \leq d - 2$ . So it suffices to prove that  $H_5^\sigma, H_6^\sigma \succ B_{n-d+s+1,s+1}^{\bar{-},\bar{-},\bar{-}}, H_7^\sigma \succ B_{n-d+s+2,s+2}^{\bar{-},\bar{-},\bar{-}}, H_8^\sigma \succ B_{n-d+s,s}^{\bar{-},\bar{-},\bar{-}}$ , where  $H_5 = G - \{x_{k-2}, \dots, x_0\} - \{x_{l+1}, \dots, x_d\}$ ,  $H_7 = G - \{x_{k-2}, \dots, x_0\} - \{x_{l+2}, \dots, x_d\}$ ,  $H_8 = G - \{x_{k-1}, \dots, x_0\} - \{x_{l+1}, \dots, x_d\}$ , and  $H_6 = G - \{x_{k-1}, \dots, x_0\} - \{x_{l+2}, \dots, x_d\}$ . Let  $d_j = d(H_j)$  and  $n_j = |V(H_j)|$ , where  $j = 5, 6, 7, 8$ . Then  $d_j \geq 4$ . If  $n_j - d_j < h$ , then by the induction hypothesis and Lemma 2.14, we have the desired result.

Suppose that  $n_j - d_j = h$ . If  $x_{k-1}$  lies on all diametrical paths of  $H_5$ , then by Lemmas 2.14 and 3.3,  $H_5^\sigma \succ B_{n-d+s+1,s+1}^{\bar{-},\bar{-},\bar{-}}$ . Otherwise, by similar arguments as those in Case 2, we also have  $H_5^\sigma \succ B_{n-d+s+1,s+1}^{\bar{-},\bar{-},\bar{-}}$ . Similarly,  $H_6^\sigma \succ B_{n-d+s+1,s+1}^{\bar{-},\bar{-},\bar{-}}$ . By Lemmas 2.14 and 3.2, we have  $H_8^\sigma \succ B_{n-d+s,s}^{\bar{-},\bar{-},\bar{-}}$ . If there exists some diametrical path  $P(H_7)$  such that  $x_{k-1}$  or  $x_{l+1}$  lies outside  $P(H_7)$ , then by similar arguments as those in Case 2, we have  $H_7^\sigma \succ B_{n-d+s+2,s+2}^{\bar{-},\bar{-},\bar{-}}$ . Otherwise, by Lemmas 2.2, 2.4, 2.11, 2.12, 2.13 and 3.3, we have  $H_7^\sigma - x_{k-1} \succ B_{n-d+s+1,s+2}^{\bar{-},\bar{-},\bar{-}}$ ,  $H_7^\sigma - x_{k-1} - x_k \succ U_{n-d+s,s}^- \succ T_{n-d+s,s} \succ T_{s+3,s}$ , and then  $H_7^\sigma \succ B_{n-d+s+2,s+2}^{\bar{-},\bar{-},\bar{-}}$ .

**Subcase 3.2.** There is only one pendent vertex. By similar arguments as those in Subcase 3.1, we have  $G^\sigma \succ B_{n,d}^{\bar{-},\bar{-},\bar{-}}$ .

Combining all those cases above, we complete the proof.  $\blacksquare$

**Theorem 3.5.** Let  $G \in \mathcal{B}(n, d)$  with  $3 \leq d \leq n-3$  and  $G \neq B_{n,d}$ . If  $t \geq 1$ , then  $G^\sigma \succ B_{n,d}^{\bar{-},\bar{-},\bar{-}}$ .

**Proof.** We prove this theorem by induction on  $n - d$ .

By Lemma 3.1, the result holds for  $n - d = 3$ . Let  $h \geq 4$  and suppose that the result holds for  $n - d < h$ . Now suppose that  $n - d = h$  and  $G \in \mathcal{B}(n, d)$ .

**Case 1.** There is no pendent vertex in  $G$ . By Lemma 3.2, we have  $G^\sigma \succ B_{n,d}^{\bar{-},\bar{-},\bar{-}}$ .

**Case 2.** There is a pendent vertex  $u$  outside some diametrical path  $P(G) = x_0x_1 \dots x_d$ . Let  $v$  be the neighbor of  $u$ . Then  $G - u \in \mathcal{B}(n - 1, d)$ . By the induction hypothesis,  $G^\sigma - u \succ B_{n-1,d}^{\bar{-},\bar{-},\bar{-}}$ . Let  $H = G - u - v$ , it suffices to prove that  $a_{2i}(H^\sigma) \geq a_{2i}(T_{d+1,d-2})$  by the (7).

**Subcase 2.1.**  $v$  lies on some cycle, say  $C_a$ .

**Subcase 2.1.1.** Suppose that  $v = u_0$  or  $u_{t-1}$ .

If  $v$  lies outside  $P(G)$ , then  $H \supseteq P(G)$ . By Lemmas 2.4 and 2.7, we have  $a_{2i}(H^\sigma) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2})$ .

If  $v$  lies on  $P(G)$ . Let  $v = x_k$ . If  $C_a$  and  $C_b$  have exactly one common vertex, then  $H \supseteq P_2 \cup P_2 \cup P_k \cup P_{d-k}$ ,  $P_2 \cup P_k \cup P_{d-k+1}$ ,  $P_2 \cup P_k \cup T_1$ ,  $P_{k+1} \cup P_{d-k+1}$ ,  $P_{k+1} \cup T_1$  or  $T_1 \cup T_2$ , where  $T_1 \in T(d - k + 1, d - k - 1)$ ,  $T_2 \in T(k + 1, k - 1)$ . If  $C_a$  and  $C_b$  have at least two common vertices,  $H \supseteq P_3 \cup P_k \cup P_{d-k}$ ,  $P_k \cup P_{d-k+2}$ ,  $P_k \cup T_3$ ,  $P_k \cup T_4$  or  $P(G)$ , where  $T_3 \in T(d - k + 2, d - k - 1)$ ,  $T_4 \in T(d - k + 2, d - k)$ .

If  $H \supseteq P_2 \cup P_2 \cup P_k \cup P_{d-k}, P_2 \cup P_k \cup P_{d-k+1}, P_{k+1} \cup P_{d-k+1}, P_3 \cup P_k \cup P_{d-k}, P_k \cup P_{d-k+2}$  or  $P(G)$ , by Lemmas 2.4, 2.6 and 2.7, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $H \supseteq P_2 \cup P_k \cup T_1$  or  $P_{k+1} \cup T_1$ , by Lemmas 2.4, 2.6, 2.8, 2.9 and 2.10, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{k+1} \cup T_{d-k+1,d-k-1}) \geq a_{2i}(T_{d+1,d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $H \supseteq T_1 \cup T_2$ , by Lemmas 2.4, 2.8 and 2.10, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(T_{k+1,k-1} \cup T_{d-k+1,d-k-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $H \supseteq P_k \cup T_3$ , by Lemmas 2.4, 2.8 and 2.10, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup T_{d-k+2,d-k-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $H \supseteq P_k \cup T_4$ , by Lemmas 2.4, 2.8, 2.9 and 2.10, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup T_{d-k+2,d-k}) \geq a_{2i}(T_{d+1,d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

**Subcase 2.1.2.** Suppose that  $v \neq u_0$  and  $u_{t-1}$ . If  $v$  lies outside  $P(G)$ , then  $H \supseteq H_1$  or  $P(G) \cup C_s$ , where  $H_1 \in U(s_1, d)$ ,  $s_1 \geq d+2$   $s = b$  or  $c$ .

If  $H \supseteq H_1$ , by Lemmas 2.4, 2.7, 2.11 and 2.12, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(H_1^\sigma) \geq a_{2i}(U_{s_1,d}^-) \geq a_{2i}(T_{s_1,d}) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $H \supseteq P(G) \cup C_s$ , by Lemmas 2.4, 2.5 and 2.7, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_{d+1} \cup S_s) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $v$  lies on  $P(G)$ , then  $P(G)$  and  $C_a$  have common vertices, say,  $x_k, \dots, x_{k+p}$ , where  $p \geq 0$ .

If  $p = 0$ , then  $k \geq 1$ ,  $H \supseteq P_k \cup P_{d-k} \cup C_b$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup P_{d-k} \cup S_b) \geq a_{2i}(P_{d-1} \cup P_3) \geq a_{2i}(P_{d+1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $p \geq 1$ . If  $v \neq x_k, x_{k+p}$ , then  $H \supseteq H_2$ , where  $H_2 \in U(s_2, d)$ ,  $s_2 \geq d+2$ . We can prove that  $a_{2i}(H^\sigma) \geq a_{2i}(T_{d+1,d-2})$  by the similar arguments as above. For  $v = x_k$  or  $v = x_{k+p}$ , say  $v = x_k$ , then  $k \geq 1$ ,  $H \supseteq P_k \cup H_3$  or  $P_k \cup H_4$ , where  $H_3 \in U(s_3, d_3)$ ,  $s_3 \geq d_3+2$ ,  $d_3 \geq d-k \geq 2$  and  $H_4$  is a graph obtained by attaching  $P_{d-k-2}$  to a vertex of  $C_3$ .

Suppose that  $H \supseteq P_k \cup H_3$ . If  $d_3 = 2$ , then  $k = d-2$ ,  $s_3 \geq 4$ . By Lemmas 2.4, 2.5, 2.9 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup S_{s_3}) \geq a_{2i}(P_{d-2} \cup T_{4,2}) \geq a_{2i}(T_{d+1,d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d_3 \geq 3$ , by Lemmas 2.4, 2.9, 2.10, 2.11 and 2.12, then we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup U_{s_3,d_3}^-) \geq a_{2i}(P_k \cup T_{s_3,d_3}) \geq a_{2i}(T_{d+1,d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

Suppose that  $H \supseteq P_k \cup H_4$ . If  $d-k-2 = 0$ , then  $k = d-2$ . We also have  $a_{2i}(C_3) \geq a_{2i}(T_{4,2})$ . By Lemmas 2.4, 2.9 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup C_3) \geq a_{2i}(P_{d-2} \cup T_{4,2}) \geq a_{2i}(T_{d+1,d-1}) \geq a_{2i}(T_{d+1,d-2}).$$

If  $d - k - 2 \geq 1$ , by Lemmas 2.2, 2.4, 2.6 and 2.8, then we have

$$\begin{aligned}
a_{2i}(H^\sigma) &\geq a_{2i}(P_k \cup H_4^* - u_0 u_1) + a_{2i-2}(P_k \cup H_4^* - u_0 - u_1) \\
&\geq a_{2i}(P_k \cup T_{d-k+1, d-k-1}) + a_{2i-2}(P_k \cup P_{d-k-1}) \\
&= a_{2i}(P_k \cup P_{d-k-1}) + 2a_{2i-2}(P_k \cup P_{d-k-2}) + a_{2i-2}(P_k \cup P_{d-k-1}) \\
&\geq a_{2i}(P_{d-2}) + 3a_{2i-2}(P_{d-3}) = a_{2i}(T_{d+1, d-2}).
\end{aligned}$$

**Subcase 2.2.**  $v$  lies outside any cycle.

**Subcase 2.2.1** Suppose that  $v$  lies on  $P(G)$ . Let  $v = x_k$ . If  $P(G)$  and any cycle have no common vertices, then  $H \supseteq C_c \cup P_k \cup P_{d-k}$ . By Lemmas 2.4, 2.5, 2.6 and 2.7, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(C_c \cup P_k \cup P_{d-k}) \geq a_{2i}(P_3 \cup P_{d-1}) \geq a_{2i}(T_{d+1, d-2}).$$

If some vertex of  $P(G)$  lies on one cycle, then  $H \supseteq P_k \cup H_1$ , where  $H_1 \in B(s_1, d_1)$ ,  $d_1 + 2 \leq s_1 \leq n - 2 - k$ ,  $d_1 \geq \max\{d - k - 1, 2\}$ . Suppose that  $s_1 \geq d_1 + 3$ . If  $d_1 = 2$ , then  $k \geq d - 3$ ,  $s_1 \geq 5$ . By Lemmas 2.4, 2.5 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup S_{s_1}) \geq a_{2i}(P_k \cup T_{5,2}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $d_1 \geq 3$ , then  $s_1 - d_1 < h$ ,  $H_1^\sigma \succ B_{s_1, d_1}^{-, -, -}$  by the induction hypothesis. So by Lemmas 2.4 and 2.10, and (6), we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup H_1^\sigma) \geq a_{2i}(P_k \cup B_{s_1, d_1}^{-, -, -}) \geq a_{2i}(P_k \cup T_{s_1, d_1}) \geq a_{2i}(T_{d+1, d-2}).$$

Suppose that  $s_1 = d_1 + 2$ . Then  $H_1$  is obtained by attaching respectively paths  $P_l$  and  $P_{d_1-l-2}$  to the two non-adjacent vertices in  $K_4 - e$ . If  $d_1 = 2$ , then  $k \geq d - 3$ . We also have  $a_{2i}((K_4 - e)^{*, *, -}) \geq a_{2i}(T_{5,2})$ , so by Lemmas 2.4 and 2.10, we have

$$a_{2i}(H^\sigma) \geq a_{2i}(P_k \cup (K_4 - e)^{*, *, -}) \geq a_{2i}(P_k \cup T_{5,2}) \geq a_{2i}(T_{d+1, d-2}).$$

If  $d_1 \geq 3$ , by Lemmas 2.2, 2.4, 2.11 and 2.12, then we have

$$\begin{aligned}
a_{2i}(H^\sigma) &\geq a_{2i}(P_k \cup H_1^{*, *, -} - u_0 u_1) + a_{2i-2}(P_k \cup H_1^{*, *, -} - u_0 - u_1) \\
&\geq a_{2i}(P_k \cup U_{s_1, d_1}^-) + a_{2i-2}(P_k \cup P_{l+1} \cup P_{d_1-l-1}) \\
&\geq a_{2i}(P_k \cup T_{d-k+1, d-k-1}) + a_{2i-2}(P_k \cup P_{d-k-2}) \geq a_{2i}(T_{d+1, d-2}).
\end{aligned}$$

**Subcase 2.2.2.** Suppose that  $v$  lies outside  $P(G)$ ,  $G \supseteq C_a \cup C_b \cup P(G)$  or  $H_1$ , where  $H_1 \in U(s_1, d)$  with  $d + 2 \leq s_2 \leq n - 2$ . We can prove  $a_{2i}(H^\sigma) \geq a_{2i}(T_{d+1, d-2})$  by similar arguments as above.

**Case 3.** All pendent vertices are contained in the  $P(G)$ . By similar arguments as those in Case 3 of Theorem 3.4,  $G^\sigma \succ B_{n,d}^{-, -, -}$ .

Combining all those cases above, we complete the proof. ■

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